The Vector Forms of Planes

Dr. William J. Larson - http://MathsTutorGeneva.ch/

Cartesian Form of the equation of a Plane
ax + by + cz = D, where a, b, c and D are scalars.
Example: 3x + 5y - 2z = 5

Normal Vector Form of the equation of a Plane

$\vec{r} \bullet \vec{n} = D$, where $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, is the position vector, $\vec{n}$ is a vector <u>perpendicular</u> to the plane and D is a scalar.

Example: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 5$.

Vector Form of the equation of a Plane

$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ or

$\vec{r} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \mu \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, where $\vec{a}$ is a position vector giving the position of a point on the plane at $\lambda = \mu = 0$, $\vec{b}$ and $\vec{c}$ are non-parallel vectors in the plane and parameters λ and μ are variables.

Example: $\vec{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

Converting between the Forms

Normal Vector Form to Cartesian Form and vice versa
Evaluation of the dot product of the vector form immediately gives the Cartesian form.

Example Convert $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 5$ to Cartesian form.

Doting gives 3x + 5y – 2z = 5.

"Undoting" the Cartesian form immediately gives the vector form.

Cartesian Form to Vector Form
Use the Cartesian form to find three points A, B & C on the plane. Use $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{AB}$ and $\vec{c} = \overrightarrow{BC}$ or whichever are convenient.
Example Convert 3x + 5y – 2z = 5 to parametric vector form.
A is (2, 1, 3), B is (0, 1, 0), C is (1, 0, -1).

$\vec{a} = \overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\vec{b} = \overrightarrow{BA} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, $\vec{c} = \overrightarrow{BC} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, so:

$\vec{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

Vector Form to Normal Vector Form
We need a vector $\vec{n}$ perpendicular to the plane, so use $\vec{n} = \vec{b} \times \vec{c}$. Then we use $\vec{a}$ to find D.

Example Convert $\vec{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ to vector form.

$\vec{n} = \vec{b} \times \vec{c} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$.

$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 5$, so our equation is $\vec{r} \bullet \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 5$.

Vector Form to Cartesian Form
Convert to vector form as above and then dot it into Cartesian form.

Example Convert $\vec{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ to Cartesian form.
From the above example we know the equation is

$\vec{r} \bullet \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 5$, which dots into 3x + 5y - 2z = 5.

Planes with Vectors Problems vs. Planes with Matrices Problems

We use both matrices and vectors to solve problems with planes. However the types of problems solvable with matrices and vectors are different.
Planes with vectors problems tend to be finding the equation of one plane perpendicular to a given vector or parallel to two vectors and passing though a given point.
Planes with matrices problems tend to be about three planes and finding the values of one or two unknown coefficients such that the planes intersect in a point, line, or not at all.

Given three points in a plane, construct the 3 equations

Call the points A, B, & C
Create 2 vectors from the 3 points, for example **AB** & **AC.**

To create the Normal Vector Form of the Equation of a Plane

n = AB × AC

Use any point, say A, to find d. d = **OA ● n**

The normal vector equation is r ● n = d

Example

A (1, 2, 3), B (2, -1, 4), C (-2, 1, 3)

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -10 \end{pmatrix}$$

$$d = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ -10 \end{pmatrix} = 1 - 6 - 30 = -35$$

So our equation is $\vec{r} \bullet \begin{pmatrix} 1 \\ -3 \\ -10 \end{pmatrix} = -35$

To create the Cartesian Form of the Equation of a Plane

Dot out the normal vector equation.

Example

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ -10 \end{pmatrix} = -35$$

So our equation is x – 3y – 10z = -35

To create the Vector Form of the Equation of a Plane

Again use $\overrightarrow{AB}, \overrightarrow{AC} \text{ and } \overrightarrow{OA}$

The vector equation is $\vec{r} = \overrightarrow{OA} + \lambda\, \overrightarrow{AB} + \mu\, \overrightarrow{AC}$

Example

So our equation is $\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$