The Vector Forms of Lines in 3-D

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Cartesian Form of the equation of a Line

$\frac{x - x_0}{x_1} = \frac{y - y_0}{y_1} = \frac{z - z_0}{z_1}$, where x_0, y_0, z_0, x_1, y_1 and z_1 are scalar constants.

(x_0, y_0, z_0) is a point on the line.

$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ is the direction vector of the line.

Example

$\frac{x-3}{4} = \frac{y-5}{6} = z+2$

Parametric Form of the equation of a Line

$x(\lambda) = x_0 + x_1 \lambda$
$y(\lambda) = y_0 + y_1 \lambda$
$z(\lambda) = z_0 + z_1 \lambda$

Where parameter λ is a variable and x_0, y_0, z_0, x_1, y_1 and z_1 are scalar constants.

(x_0, y_0, z_0) is a point on the line.

$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ is the direction vector of the line.

Example

$x(t) = 3 + 4\lambda$
$y(t) = 5 + 6\lambda$
$z(t) = -2 + \lambda$

Vector Form of the equation of a Line

$\vec{r} = \vec{a} + \lambda \vec{b}$

or

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, where $\vec{a}$ is the position vector giving the position of a point on the line at λ = 0, $\vec{b}$ is a vector <u>parallel</u> to the line (the direction vector of the line) and parameter λ is a variable. (x_0, y_0, z_0) is a point on the line.

Two lines are parallel if their direction vectors are scalar multiples of each other. In other words $\vec{r}_1 = \vec{p}_1 + \lambda \vec{d}_1$ and $\vec{r}_2 = \vec{p}_2 + \mu \vec{d}_2$ are parallel, if

$\vec{d}_1 = k\,\vec{d}_2 \quad for \ \ some \ \ scalar \ \ k$.

Example

$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix}$

Converting Between the Forms

Cartesian Form to Parametric Form and vice versa

To convert from Cartesian form to parametric form, let each term equal λ and solve for x, y & z.

Example: Convert $\frac{x-3}{4} = \frac{y-5}{6} = z+2$ to parametric form.

Let $\lambda = \frac{x-3}{4} = \frac{y-5}{6} = z+2$.

Therefore
$x = 4\lambda + 3$,
$y = 6\lambda + 5$,
$z = \lambda - 2$.

To convert from parametric form to Cartesian form, solve each equation for λ and set the three resulting equations equal.

Parametric Form to Vector Form and vice versa

This is easy since it just requires adding or removing vector brackets.

Example: Convert
$x = 4\lambda + 3$,
$y = 6\lambda + 5$,
$z = \lambda - 2$
to parametric vector form.

$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix}$

Cartesian Form to Vector Form and vice versa

It is easy to convert either of these forms to parametric form and then to the other form, but one can just convert directly.

$\frac{x - x_0}{x_1} = \frac{y - y_0}{y_1} = \frac{z - z_0}{z_1}$ is the same as

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

Example:

$\frac{x-3}{4} = \frac{y-5}{6} = z+2$

is the same as

$\vec{r} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix}$.

Careful! The coefficients of the variables must be 1.

Example:

$\frac{3-2x}{4} = \frac{y-5}{6} = z+2$

is the same as

$\vec{r} = \begin{pmatrix} 3/2 \\ 5 \\ -2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix}$