Counting Principles 2008-2014 HL with MS

1a. *[3 marks]*

Fifteen boys and ten girls sit in a single line.

In how many ways can they be seated in a single line so that the boys and girls are in two separate groups?

1b. *[3 marks]*

Two boys and three girls are selected to go the theatre. In how many ways can this selection be made?

2a. *[3 marks]*

On Saturday, Alfred and Beatrice play 6 different games against each other. In each game, one of the two wins. The probability that Alfred wins any one of these games is $\frac{2}{3}$.

Show that the probability that Alfred wins exactly 4 of the games is $\frac{80}{243}$.

2b. *[4 marks]*

(i) Explain why the total number of possible outcomes for the results of the 6 games is 64.

(ii) By expanding $(1+x)^6$ and choosing a suitable value for *x*, prove

$$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$$

(iii) State the meaning of this equality in the context of the 6 games played.

2c. *[9 marks]*

The following day Alfred and Beatrice play the 6 games again. Assume that the probability that Alfred wins any one of these games is still $\frac{2}{3}$.

(i) Find an expression for the probability Alfred wins 4 games on the first day and 2 on the second day. Give your answer in the form $\binom{6}{r}^2 \left(\frac{2}{3}\right)^s \left(\frac{1}{3}\right)^t$ where the values of *r*, *s* and *t* are to be found.

(ii) Using your answer to (c) (i) and 6 similar expressions write down the probability that Alfred wins a total of 6 games over the two days as the sum of 7 probabilities.

(iii) Hence prove that $\binom{12}{6} = \binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2$.

2d. *[6 marks]*

Alfred and Beatrice play *n* games. Let *A* denote the number of games Alfred wins. The expected value of *A* can be written as $\mathrm{E}(A) = \sum_{r=0}^{n} r \binom{n}{r} \frac{a^r}{b^n}$.

(i) Find the values of *a* and *b*.

(ii) By differentiating the expansion of $(1+x)^n$, prove that the expected number of games Alfred wins is $\frac{2n}{3}$.

3a. *[3 marks]*

Three boys and three girls are to sit on a bench for a photograph.

Find the number of ways this can be done if the three girls must sit together.

3b. *[4 marks]*

Find the number of ways this can be done if the three girls must all sit apart.

4a. *[4 marks]*

(i) Express the sum of the first *n* positive odd integers using sigma notation.

(ii) Show that the sum stated above is n^2.

(iii) Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers.

4b. *[7 marks]*

A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points.

(i) Show on a diagram all diagonals if there are 5 points.

(ii) Show that the number of diagonals is $\frac{n(n-3)}{2}$ if there are n points, where $n > 2$.

(iii) Given that there are more than one million diagonals, determine the least number of points for which this is possible.

5. *[5 marks]*

Six people are to sit at a circular table. Two of the people are not to sit immediately beside each other. Find the number of ways that the six people can be seated.

6. *[4 marks]*

Find the number of ways in which seven different toys can be given to three children, if the youngest is to receive three toys and the others receive two toys each.

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1a. *[3 marks]*

Markscheme

number of arrangements of boys is $15!$ and number of arrangements of girls is $10!$ ***(A1)***

total number of arrangements is $15! \times 10! \times 2 (= 9.49 \times 10^{18})$ ***M1A1***

Note: If 2 is omitted, award ***(A1)M1A0***.

[3 marks]

Examiners report

A good number of correct answers were seen to this question, but a significant number of candidates forgot to multiply by 2 in part (a) and in part (b) the most common error was to add the combinations rather than multiply them.

1b. *[3 marks]*

Markscheme

number of ways of choosing two boys is $\binom{15}{2}$ and the number of ways of choosing three girls is $\binom{10}{3}$ ***(A1)***

number of ways of choosing two boys and three girls is $\binom{15}{2} \times \binom{10}{3} = 12600$ ***M1A1***

[3 marks]

Examiners report

A good number of correct answers were seen to this question, but a significant number of candidates forgot to multiply by 2 in part (a) and in part (b) the most common error was to add the combinations rather than multiply them.

2a. *[3 marks]*

Markscheme

$B\left(6, \frac{2}{3}\right)$ ***(M1)***

$p(4) = \binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2$ ***A1***

$\binom{6}{4} = 15$ ***A1***

$= 15 \times \frac{2^4}{3^6} = \frac{80}{243}$ ***AG***

[3 marks]

Examiners report

This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

(a) Candidates need to be aware how to work out binomial coefficients without a calculator

2b. *[4 marks]*

Markscheme

(i) 2 outcomes for each of the 6 games or $2^6 = 64$ ***R1***

(ii) $(1+x)^6 = \binom{6}{0} + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 + \binom{6}{5}x^5 + \binom{6}{6}x^6$ ***A1***

Note: Accept ${}^nC_r$ notation or $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$

setting $x = 1$ in both sides of the expression ***R1***

Note: Do not award ***R1*** if the right hand side is not in the correct form.

$$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$$ ***AG***

(iii) the total number of outcomes = number of ways Alfred can win no games, plus the number of ways he can win one game *etc.* ***R1***

[4 marks]

Examiners report

This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

(b) (ii) A surprising number of candidates chose to work out the values of all the binomial coefficients (or use Pascal's triangle) to make a total of 64 rather than simply putting 1 into the left hand side of the expression.

2c. *[9 marks]*

Markscheme

(i) Let $\mathrm{P}(x,\ y)$ be the probability that Alfred wins *x* games on the first day and *y* on the second.

$$\mathrm{P}(4,2) = \binom{6}{4} \times \left(\tfrac{2}{3}\right)^4 \times \left(\tfrac{1}{3}\right)^2 \times \binom{6}{2} \times \left(\tfrac{2}{3}\right)^2 \times \left(\tfrac{1}{3}\right)^4$$ ***M1A1***

$\binom{6}{2}^2 \left(\tfrac{2}{3}\right)^6 \left(\tfrac{1}{3}\right)^6$ or $\binom{6}{4}^2 \left(\tfrac{2}{3}\right)^6 \left(\tfrac{1}{3}\right)^6$ ***A1***

$r = 2$ or 4, $s = t = 6$

(ii) P(Total = 6) =

P(0, 6) + P(1, 5) + P(2, 4) + P(3, 3) + P(4, 2) + P(5, 1) + P(6, 0) ***(M1)***

$$= \binom{6}{0}^2 \left(\tfrac{2}{3}\right)^6 \left(\tfrac{1}{3}\right)^6 + \binom{6}{1}^2 \left(\tfrac{2}{3}\right)^6 \left(\tfrac{1}{3}\right)^6 + \ldots + \binom{6}{6}^2 \left(\tfrac{2}{3}\right)^6 \left(\tfrac{1}{3}\right)^6$$ ***A2***

$$= \frac{2^6}{3^{12}}\left(\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2\right)$$

Note: Accept any valid sum of 7 probabilities.

(iii) use of $\binom{6}{i} = \binom{6}{6-i}$ ***(M1)***

(can be used either here or in (c)(ii))

P(wins 6 out of 12) $= \binom{12}{6} \times \left(\tfrac{2}{3}\right)^6 \times \left(\tfrac{1}{3}\right)^6 = \frac{2^6}{3^{12}}\binom{12}{6}$ ***A1***

$$= \frac{2^6}{3^{12}}\left(\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2\right) = \frac{2^6}{3^{12}}\binom{12}{6}$$ ***A1***

therefore $\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 = \binom{12}{6}$ ***AG***

[9 marks]

Examiners report

This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

2d. *[6 marks]*

Markscheme

(i) $\mathrm{E}(A) = \sum_{r=0}^{n} r \binom{n}{r} \left(\frac{2}{3}\right)^r \left(\frac{1}{3}\right)^{n-r} = \sum_{r=0}^{n} r \binom{n}{r} \frac{2^r}{3^n}$

(a = 2, b = 3) ***M1A1***

Note: ***M0A0*** for a = 2, b = 3 without any method.

(ii) $n(1+x)^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r x^{r-1}$ ***A1A1***

(sigma notation not necessary)

(if sigma notation used also allow lower limit to be r = 0)

let x = 2 ***M1***

$n3^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r 2^{r-1}$

multiply by 2 and divide by 3^n ***(M1)***

$\frac{2n}{3} = \sum_{r=1}^{n} \binom{n}{r} r \frac{2^r}{3^n} \left(= \sum_{r=0}^{n} \binom{n}{r} \frac{2^r}{3^n}\right)$ ***AG***

[6 marks]

Examiners report

This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

(d) This was poorly done. Candidates were not able to manipulate expressions given using sigma notation.

3a. *[3 marks]*

Markscheme

the three girls can sit together in 3! = 6 ways ***(A1)***

this leaves 4 ‘objects’ to arrange so the number of ways this can be done is 4! ***(M1)***

so the number of arrangements is $6 \times 4! = 144$ ***A1***

[3 marks]

Examiners report

Some good solutions to part (a) and certainly fewer completely correct answers to part (b). Many candidates were able to access at least partial credit, if they were showing their reasoning.

3b. *[4 marks]*

Markscheme

Finding more than one position that the girls can sit ***(M1)***

Counting exactly four positions ***(A1)***

number of ways $= 4 \times 3! \times 3! = 144$ ***M1A1 N2***

[4 marks]

Examiners report

Some good solutions to part (a) and certainly fewer completely correct answers to part (b). Many candidates were able to access at least partial credit, if they were showing their reasoning.

4a. *[4 marks]*

Markscheme

(i) $\sum_{k=1}^{n}(2k-1)$ (or equivalent) ***A1***

Note: Award ***A0*** for $\sum_{n=1}^{n}(2n-1)$ or equivalent.

(ii) **EITHER**

$2 \times \frac{n(n+1)}{2} - n$ ***M1A1***

OR

$\frac{n}{2}(2+(n-1)2)$ $\left(\text{using } S_n = \frac{n}{2}(2u_1+(n-1)d)\right)$ ***M1A1***

OR

$\frac{n}{2}(1+2n-1)$ $\left(\text{using } S_n = \frac{n}{2}(u_1+u_n)\right)$ ***M1A1***

THEN

$= n^2$ ***AG***

(iii) $47^2 - 14^2 = 2013$ ***A1***

[4 marks]

Examiners report

In part (a) (i), a large number of candidates were unable to correctly use sigma notation to express the sum of the first *n* positive odd integers. Common errors included summing $2n-1$ from 1 to *n* and specifying sums with incorrect limits. Parts (a) (ii) and (iii) were generally well done.

4b. *[7 marks]*

Markscheme

(i) **EITHER**

a pentagon and five diagonals ***A1***

OR

five diagonals (circle optional) ***A1***

(ii) Each point joins to *n* – 3 other points. ***A1***

a correct argument for $n(n-3)$ ***R1***

a correct argument for $\frac{n(n-3)}{2}$ ***R1***

(iii) attempting to solve $\frac{1}{2}n(n-3) > 1\,000\,000$ for *n*. ***(M1)***

$n > 1415.7$ ***(A1)***

$n = 1416$ ***A1***

[7 marks]

Examiners report

Parts (b) (i) and (iii) were generally well done. In part (b) (iii), many candidates unnecessarily simplified their quadratic when direct GDC use could have been employed. A few candidates gave $n > 1416$ as their final answer. While some candidates displayed sound reasoning in part (b) (ii), many candidates unfortunately adopted a ‘proof by example’ approach.

5. *[5 marks]*

Markscheme

EITHER

with no restrictions six people can be seated in $5! = 120$ ways ***A1***

we now count the number of ways in which the two restricted people will be sitting next to each other

call the two restricted people p_1 and p_2

they sit next to each other in two ways ***A1***

the remaining people can then be seated in $4!$ ways ***A1***

the six may be seated p_1 and p_2 next to each other) in $2 \times 4! = 48$ ways ***M1***

$\therefore$ with p_1 and (p_2 not next to each other the number of ways $= 120 - 48 = 72$ ***A1 N3***

[5 marks]

OR

person p_1 seated at table in 1 way ***A1***

p_2 then sits in any of 3 seats (not next to p_1) ***M1A1***

the remaining 4 people can then be seated in $4!$ ways ***A1***

$\therefore$ number ways with p_1 not next to p_2 $= 3 \times 4! = 72$ ways ***A1 N3***

Note: If candidate starts with $6!$ instead of $5!$, potentially leading to an answer of 432, do not penalise.

[5 marks]

Examiners report

Very few candidates provided evidence of a clear strategy for solving such a question. The problem which was set in a circular scenario was no more difficult than an analogous linear one.

6. *[4 marks]*

Markscheme

the number of ways of allocating presents to the first child is $\binom{7}{3} \left(\text{or} \binom{7}{2}\right)$ ***(A1)***

multiplying by $\binom{4}{2} \left(\text{or} \binom{5}{3} \text{ or} \binom{5}{2}\right)$ ***(M1)(A1)***

Note: Award ***M1*** for multiplication of combinations.

$\binom{7}{3}\binom{4}{2} = 210$ ***A1***

[4 marks]