Inverse Functions

Dr. W. Larson, International School of Geneva

An inverse function "undoes" whatever its corresponding function does.

The inverse of the inverse is the original function.

A function "undoes" whatever its corresponding inverse function does.

$f^{-1}(x)$ is **pronounced** "*f* inverse of *x*".

Careful, the symbol is confusing:

$f^{-1}(x) \neq (f(x))^{-1} = \frac{1}{f(x)}$ = the **reciprocal** of $f(x)$.

Examples:

$f(x)$ = Function	$f^{-1}(x)$ = Inverse Function*
x^2	$\sqrt{x}$
x^3	$\sqrt[3]{x}$
$x + 1$	$x - 1$
$3x$	$\frac{x}{3}$
$\sin x$	$\sin^{-1}x \equiv \arcsin x$
b^x	$\log_b x$
10^x	$\log x$
e^x	$\ln x$

*One could just as well reverse any pair in the above two columns, i.e. listing ln x as the function & e^x as the inverse function.

Finding the Inverse of *f*(*x*)

1. Replace $f(x)$ by y.
2. Interchange x & y.
3. Solve for y.
4. Replace y by $f^{-1}(x)$.

Definition of the Inverse Function

Algebraically the definition of the inverse function is: $f^{-1}(f(x)) = x$ or $f(f^{-1}(x)) = x$.

Graphically the definition of the inverse function is that the function and its inverse are reflections of each other about the line $y = x$.

The properties of *x* and *y* are exchanged:

For $f(x)$ & $f^{-1}(x)$:

1. The y-intercept of one is the x-intercept of the other.
2. The domain of one is the range of the other.
3. The vertical asymptotes of one are the horizontal asymptotes of the other.
4. If their graphs cross, they must cross on the line $y = x$.

Horizontal Line Test

A function can have an inverse **function** only if passes the Horizontal Line Test. If a **horizontal** line intersects the graph of $f(x)$ in more than one point, then $f(x)$ fails the HLT.

Remember that the VLT does something else. The VLT determines whether a relation is a function.

Restricting the domain

If a function fails the horizontal line test, then it does not have an inverse **function**. However by restricting the domain of $f(x)$, it is usually possible to construct a new function which does pass the horizontal line test &and thus does have an inverse function.

Example $y = x^2$ fails the HLT, so we restrict its domain to $x \geq 0$. Then it **does** have an inverse.

One-to-one, etc.

"Many" below means "two or more."

A relation is **one-to-one** if for every input, there is only one output and for every output, there is only one input.

A relation is **one-to-many,** if one input produces many outputs.

A relation is **many-to-one,** if many inputs produce the same one output.

A relation is **many-to-many,** one input produces many outputs and if many inputs produce the same one output.

Relations which are one-to-many or many-to-many fail the vertical line test and so are **not functions**.

If the relation is not a function, we do not proceed to try to construct its inverse.

Relations which are one-to-one or many-to-one pass the vertical line test and so are **functions**.

A relation that is **many-to-one** fails the horizontal line test. So its inverse is not a function. So to construct its inverse we must first restrict its domain.

A function can have an inverse **function** only if it is one-to-one.

Passing both the **horizontal line test and the vertical line test** is equivalent to being one-to-one.

The IB uses the phrase "one-one" rather than "one-to-one" etc.