

Vertical and Horizontal Asymptotes of Rational Functions

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An **asymptote** is a line that the graph of an equation approaches, but never reaches.

A **rational function** is a polynomial divided by another polynomial.

Vertical Asymptotes

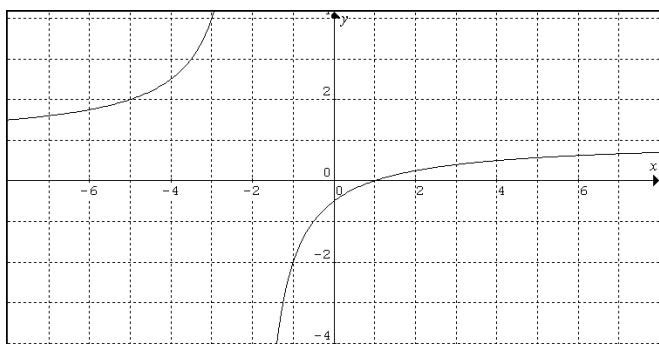
A function approaches either $+\infty$ or $-\infty$ as x approaches a vertical asymptote.

There is a vertical asymptote at those values of x that make the denominator equal to zero.

Example $f(x) = \frac{x-1}{x+2}$.

There is a vertical asymptote if $x + 2 = 0$.

Therefore the line $x = -2$ is a vertical asymptote.



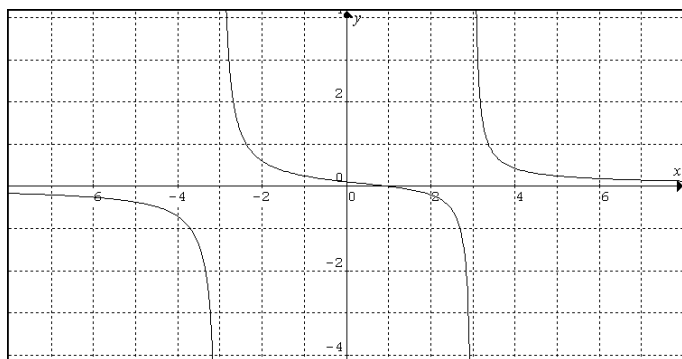
Example $f(x) = \frac{x-1}{x^2-9}$.

There is a vertical asymptote if $x^2 - 9 = 0$.

So there is a vertical asymptote if

$$(x-3)(x+3) = 0.$$

Therefore the lines $x = 3$ and $x = -3$ are vertical asymptotes.



The graph of a function **cannot** cross a vertical asymptote, because if it did, then it would not be a function.

Horizontal Asymptotes

A horizontal asymptote is a horizontal line that the function approaches as x approaches $+\infty$ or $-\infty$.

A horizontal asymptote is the limit of $y = f(x)$ as x approaches

$$+\infty \text{ or } -\infty, \text{ i.e. } y = \lim_{x \rightarrow \infty} f(x).$$

A rational function is of the form

$$f(x) = \frac{a_n x^n + \dots + a_1 x^1 + a_0}{b_m x^m + \dots + b_1 x^1 + b_0}$$

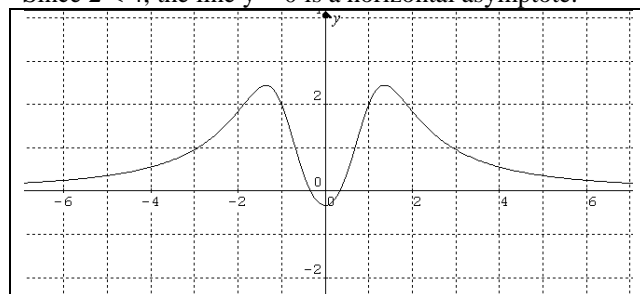
To investigate horizontal asymptotes, the only thing you need to consider is the relative values of **n** & **m**, the highest powers in the numerator and denominator respectively.

There are three cases:

1. If the power in the denominator is bigger, then the x -axis ($y = 0$) is a horizontal asymptote.

Example $f(x) = \frac{9x^2 - 1}{x^4 + 3}$. HA: $y = \lim_{x \rightarrow \infty} f(x) = 0$

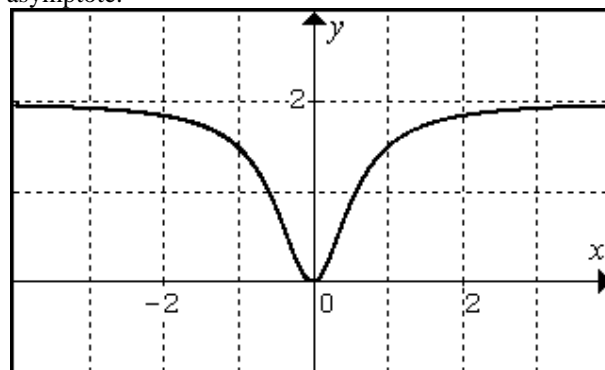
Since $2 < 4$, the line $y = 0$ is a horizontal asymptote.



2. If the powers are the same, then there is a horizontal asymptote at y equals the ratios of the coefficients of the highest power terms in the numerator and denominator.

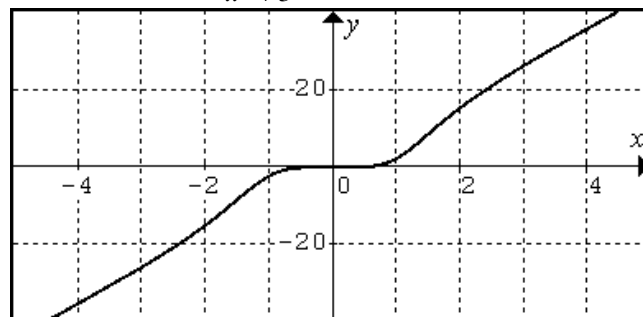
Example $f(x) = \frac{6x^2}{3x^2 + 1}$. HA: $y = \lim_{x \rightarrow \infty} f(x) = 2$

Since $2 = 2$, the line $y = \frac{6}{3}$, i.e. $y = 2$ is a horizontal asymptote.



3. If the power in the numerator is bigger, there are no horizontal asymptotes.

Example $f(x) = \frac{9x^5 - 1}{x^4 + 3}$.



The graph of a function **can** cross a horizontal asymptote. See,

for example, the graph of $f(x) = \frac{9x^2 - 1}{x^4 + 3}$ above.